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LETTER TO THE EDITOR

On chaos in spin glasses

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Abstract. The effect of field or temperature changes on spin correlations is calculated for the finite-range Ising spin glass in the Gaussian approximation. Correlation overlaps are found to decay exponentially, with characteristic lengths $\xi_H \sim H^{-2/3}$ and $\xi_{\Delta H} \sim (\Delta H)^{-1/2} H^{-1/6}$ near T_c, when the field is changed from 0 to H and from H to $H+\Delta H$, respectively, but fall off like a power for any temperature change below T_c in zero field.

Typical spin-glass effects such as remanence, hysteresis, history dependence, long relaxation, etc, are usually interpreted, mostly in a qualitative manner, in terms of a complicated free-energy landscape whose features are sensitive to small variations of the external parameters like the temperature T or the magnetic field H. A recently introduced phenomenological scaling theory (Fisher and Huse 1986, 1988, Bray and Moore 1987a) has given a more concrete formulation to the ideas about this sensitivity and has led to the recognition of what has been termed (Bray and Moore 1987b) the chaotic nature of the spin-glass state. Chaos in this context means that the frozen random pattern characterising the spin glass will be completely reorganised by any change of T or H, so, for example, the overlap of local magnetisations between two systems, one in a field H, the other in zero field, is

$$\hat{q}_H = \overline{\langle S_i \rangle_H \langle S_i \rangle_0} = 0 \qquad (1)$$

or between systems at temperature T, respectively $T+\Delta T$, is

$$\hat{q}_{\Delta T} = \overline{\langle S_i \rangle_T \langle S_i \rangle_{T+\Delta T}} = 0 \qquad (2)$$

for any small H or ΔT ($\langle \ldots \rangle$ means thermal average; the bar means average over the random couplings). Another aspect of chaos is that the correlation functions

$$C_H(\boldsymbol{r}) = \overline{\langle S_i S_j \rangle_H \langle S_i S_j \rangle_0} \qquad (3)$$

$$C_{\Delta T}(\boldsymbol{r}) = \overline{\langle S_i S_j \rangle_T \langle S_i S_j \rangle_{T+\Delta T}} \qquad (4)$$

fall off to zero at long distances $r = |\boldsymbol{r}_i - \boldsymbol{r}_j| \to \infty$ with finite characteristic lengths ξ_H, respectively $\xi_{\Delta T}$, that diverge for $H \to 0$, $\Delta T \to 0$. On scales exceeding these lengths the structures in the $H, 0$, respectively T, $T+\Delta T$, ensembles completely decorrelate. This sensitivity of the spin-glass order to small changes in the control parameters is in marked contrast to the behaviour of conventional ordered systems.

The phenomenological scaling theory also implies that at the spin-glass transition the phase space splits only to two 'valleys', related by an overall reflection (Fisher and Huse 1987, Bray and Moore 1987a). This is clearly in conflict with the intricate phase

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space structure envisaged by Parisi's mean-field theory (see Mézard *et al* (1987) for a detailed exposition). In some respect, however, there may be considerable overlap between these two theories, e.g. both predict a massless phase, infinite spin-glass susceptibility and power-law-like correlations throughout the spin-glass region (Fisher and Huse (1986), Bray and Moore (1987a) on the one hand, and Sompolinsky and Zippelius (1983), Goltsev (1984a), Kondor and De Dominicis (1986) on the other), and it is evidently important to clarify their relationship as fully as possible. The problem of chaos is one of the possible points of contact.

In fact, (1) and (2) can be easily derived also within the framework of mean-field theory: (1) is implied by a result in Parisi (1983), while (2) is quoted by Binder and Young (1986) from an unpublished work by Sompolinsky. The purpose of the present letter is to look into the other aspect of chaos and calculate the correlation functions (3), (4) in a Gaussian approximation around mean-field theory. To this end we consider a standard Edwards-Anderson (1975) model for Ising spins with a finite-range interaction and assume that the dimension d of the system is high enough (presumably it must be higher than 6, see Temesvàri *et al* (1988) for a recent discussion of the problem) so that the phase space structure is similar to that in mean-field theory (many valleys with a hierarchical organisation). The main result we find under this assumption is that the Gaussian approximation to the correlation function (3) is, indeed, short ranged in the entire spin-glass region, with a characteristic length $\xi_H \sim H^{-2/3}$ near T_c but, contrary to expectations, the overlap (4) between the spin correlations at two different temperatures, T and T', is infinitely long ranged for any $\Delta T = T' - T$, as long as both T and T' are below T_c. In other words, we find 'less chaos' with respect to temperature changes than to changes in the field.

In the following we give some details of the calculation of the magnetic overlaps: the case of different temperatures requires only minor modifications. To calculate quantities like (1) or (3) one can use a slightly modified version of the standard replica trick (Edwards and Anderson 1975): the first m replicas will be considered to be in zero field, the remaining $n-m$ ones in field H; in the replica limit both n and m go to zero.

The mean-field value of the overlap $\hat{q}_H$ in (1) is determined by the (stable) extremum with respect to the order parameters of the functional (basically the free energy of the compound system):

$$f = \frac{1}{n}\Bigg\{\frac{1}{2T^2}\Bigg(\sum_{\alpha<\beta\leqslant m} q_{\alpha\beta}^2 + \sum_{\alpha\leqslant m<\beta\leqslant n} \hat{q}_{\alpha\beta}^2 + \sum_{m<\alpha<\beta\leqslant n} Q_{\alpha\beta}^2\Bigg)$$
$$- \ln \operatorname{Tr} \exp\Bigg[\sum_{\alpha>m}\frac{H}{T}S_\alpha + \frac{1}{T^2}\Bigg(\sum_{\alpha<\beta\leqslant m} q_{\alpha\beta}S_\alpha S_\beta$$
$$+ \sum_{\alpha\leqslant m<\beta\leqslant n}\hat{q}_{\alpha\beta}S_\alpha S_\beta + \sum_{m<\alpha<\beta\leqslant n} Q_{\alpha\beta}S_\alpha S_\beta\Bigg)\Bigg]\Bigg\}. \tag{5}$$

The order parameters $q_{\alpha\beta}$ and $Q_{\alpha\beta}$ refer to the $H=0$ and $H=$ *finite* ensembles, respectively. Stationarity with respect to the mixed order parameters $\hat{q}_{\alpha\beta}$ demands

$$\hat{q}_{\alpha\beta} = \langle S_\alpha S_\beta\rangle_{\mathrm{MF}} \qquad \alpha\leqslant m<\beta \tag{6}$$

where $\langle\ldots\rangle_{\mathrm{MF}}$ means average with the weight under the ln Tr in (5). It is evident that one can always choose $\hat{q}_{\alpha\beta}=0$, $\forall\alpha\leqslant m<\beta$. For this choice the spins S_α, $\alpha\leqslant m$, and S_β, $\beta>m$, are not coupled by (5); hence $\langle S_\alpha S_\beta\rangle_{\mathrm{MF}} = \langle S_\alpha\rangle_{\mathrm{MF}}\langle S_\beta\rangle_{\mathrm{MF}}$, $\alpha\leqslant m<\beta$, but the field acts only on the replicas with $\beta>m$, so $\langle S_\alpha\rangle_{\mathrm{MF}}=0$, $\alpha\leqslant m$, hence $\langle S_\alpha S_\beta\rangle_{\mathrm{MF}}=0$,

$\alpha \leqslant m < \beta$, and $\hat{q}_{\alpha\beta} = 0$ is a solution of the stationarity conditions (6) indeed. On the other hand, $\hat{q}_{\alpha\beta}$ is nothing but the overlap $\hat{q}_H$ defined in (1), so the statement $\hat{q}_H = 0$ is trivially recovered.

We note in passing that near T_c (=1 in the units used here) where one can expand (5) in the q and work out f as an explicit functional of the order parameters, one can also find another, non-zero, solution for $\hat{q}_H$. It can be shown, however, that a non-zero $\hat{q}_H$ leads to a negative mass gap in the correlation function (3); hence it is unacceptable.

With the solution $\hat{q}_{\alpha\beta} = 0$, $\alpha \leqslant m < \beta$, the $q_{\alpha\beta}$ and $Q_{\alpha\beta}$ problems decouple and we choose the standard ($H = 0$, respectively $H \neq 0$) Parisi solutions for them.

Now we consider quadratic (free, Gaussian) fluctuations about the mean-field stationary point just determined. The procedure by which one goes over from mean field to the case of finite-range forces is explained e.g. in the review by Binder and Young (1986) and will not be dwelt upon here. The essential point is that the spectrum of Gaussian fluctuations is determined by the Hessian $\mathbb{M}$, the matrix of the second derivatives of (5) with respect to the order parameters. Because of $\hat{q}_{\alpha\beta} = 0$, $\mathbb{M}$ has a block diagonal structure, and the fluctuations in q, $\hat{q}$ and Q decouple. Fluctuations about the standard Parisi solutions q, Q have been the subject of detailed investigations (De Dominicis and Kondor 1983, 1984, Kondor and De Dominicis 1983, Sompolinsky and Zippelius 1983, Goltsev 1983, 1984b, Temesvàri *et al* 1988). Our concern here is the study of fluctuations of the fields $\hat{q}_{\alpha\beta}$ about their zero average value. The corresponding block of the Hessian is given by

$$nT^2 \frac{\partial^2}{\partial \hat{q}_{\alpha\beta} \partial \hat{q}_{\gamma\delta}} f = \hat{M}_{\alpha\beta,\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{1}{T^2}\left(\langle S_\alpha S_\beta S_\gamma S_\delta \rangle_{\mathrm{MF}} - \langle S_\alpha S_\beta \rangle_{\mathrm{MF}} \langle S_\gamma S_\delta \rangle_{\mathrm{MF}} \right)$$
$$\alpha \leqslant m < \beta \qquad \gamma \leqslant m < \delta. \tag{7}$$

Since S_α, S_γ and S_β, S_δ are independent, this is equal to

$$\hat{M}_{\alpha\beta,\gamma\delta} = \left(1 - \frac{1}{T^2}\right)\delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{1}{T^2}[q_{\alpha\gamma}\delta_{\beta\delta}(1-\delta_{\alpha\gamma})$$
$$+ Q_{\beta\delta}\delta_{\alpha\gamma}(1-\delta_{\beta\delta}) + q_{\alpha\gamma}Q_{\beta\delta}(1-\delta_{\alpha\gamma})(1-\delta_{\beta\delta})]. \tag{8}$$

In the Gaussian approximation the Fourier transform of (3)

$$\hat{G}_H(\boldsymbol{p}) = \frac{1}{N}\sum_{ij} \overline{\langle S_i S_j \rangle_H \langle S_i S_j \rangle_0}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}(\boldsymbol{r}_i - \boldsymbol{r}_j)} \tag{9}$$

can be expressed in terms of $\hat{\mathbb{M}}$ as

$$\hat{G}_H(\boldsymbol{p}) = T^2 \lim_{n,m=0} \frac{1}{m(n-m)} \operatorname{Tr} \frac{1}{p^2 + \hat{\mathbb{M}}}. \tag{10}$$

Now the matrix (8) can be diagonalised relatively easily, so the trace of its resolvent can be calculated. The result is

$$\hat{G}_H(\boldsymbol{p}) = \int_0^{q_1} \mathrm{d}q \int_{Q_0}^{Q_1} \mathrm{d}Q \frac{p^2 + 1 + \chi(q)\chi(Q)}{(p^2 + 1 - \chi(q)\chi(Q))^3}$$
$$+ T\chi(Q_1) \int_0^{q_1} \frac{\mathrm{d}q}{(p^2 + 1 - \chi(q)\chi(Q_1))^2}$$
$$+ T\chi(q_1) \int_{Q_0}^{Q_1} \frac{\mathrm{d}Q}{(p^2 + 1 - \chi(q_1)\chi(Q))^2} + T^2 \frac{1}{p^2 + 1 - \chi(q_1)\chi(Q_1)} \tag{11}$$

where 0 and q_1 are the minimum and the maximum of Parisi's order parameter function

$q(x)$ in zero field; Q_0, Q_1 have the same meaning in field H,

$$\chi(q)=\frac{1}{T}\left(1-q_1+\int_q^{q_1} \mathrm{d}u\, x(u)\right) \tag{12}$$

is the 'long time' (Sompolinsky 1981), $H=0$, susceptibility, with $x(q)$ the inverse of $q(x)$, and $\chi(Q)$ is the same in field H.

Equation (11) displays the full spectrum of fluctuations of the $\hat{q}$ fields

$$\lambda_{q,Q}=1-\chi(q)\chi(Q) \qquad 0\leqslant q\leqslant q_1 \qquad Q_0\leqslant Q\leqslant Q_1. \tag{13}$$

The smallest eigenvalue corresponds to $q=0$, $Q=Q_0$. Since $\chi(0)=1$, $\chi(Q_0)<1$ (Sompolinsky 1981), we find that min $\lambda_{q,Q}>0$, i.e. the correlation function $\hat{G}_H(\boldsymbol{r})$, equation (3), falls of with a finite characteristic length $\xi_H=(\min \lambda_{q,Q})^{-1/2}$. For small H and $T\leqslant T_c$, $1-\chi(Q_0)=Q_0^2=(\frac{3}{4}H^2)^{2/3}$, so ξ_H scales with the field as $\xi_H\sim H^{-2/3}$. A tentative comparison of this with the prediction of the phenomenological theory (Bray and Moore 1987a) gives for the spin-glass stiffness exponent, $\theta=(d-3)/2$, a fairly reasonable value.

The extension of the above calculation to the case of two different fields, H_1 and H_2, applied to the two sets of replicas is straightforward, though the diagonalisation of the Hessian can now be carried out easily only in the vicinity of T_c.

The overlap

$$\hat{q}_{H_1,H_2}=\overline{\langle S_i\rangle_{H_1}\langle S_i\rangle_{H_2}} \tag{14}$$

is, of course, finite now, but the correlation function

$$C_{H_1,H_2}(\boldsymbol{r})=\overline{\langle S_iS_j\rangle_{H_1}\langle S_iS_j\rangle_{H_2}} \tag{15}$$

still decays with a finite characteristic length unless $H_2=\pm H_1$. For $H_1=H$, $H_2=H+\Delta H$, $\Delta H\ll H$ this characteristic length is given by

$$\xi_{H,H+\Delta H}\sim H^{-1/6}(\Delta H)^{-1/2}.$$

For $\Delta H\sim H$ the previous result is recovered.

Now we turn to the overlap between ensembles at two different temperatures T, T'. The relevant 'free-energy' functional is obtained from (5) by setting $H=0$ and replacing the common $1/T^2$ factors by $1/T^2$, $1/TT'$ and $1/T'^2$ in the q, $\hat{q}$ and Q terms, respectively. The solution $\hat{q}_{\alpha\beta}=0$, $\alpha\leqslant m<\beta$, of the stationarity conditions is now unique; $q_{\alpha\beta}$ and $Q_{\alpha\beta}$ are Parisi matrices corresponding to temperatures T and T'. The Hessian in the $\hat{q}$ sector is given by (8) with T^2 replaced by TT'. The Fourier transform of (4), $\hat{G}_{\Delta T}(\boldsymbol{p})$, is given by a formula very similar to (11), with the following modifications: Q_0 is set to zero, T in front of the second term is to be replaced by T', T^2 in the last term by TT', and $\chi(q)$, $\chi(Q)$ are now the zero field susceptibilities corresponding to T and T', respectively.

The spectrum is still given by (13), with the new meaning of the χ. Since, for $H=0$, $\chi(0)=1$ is an identity for all $T<T_c$, we see immediately that the smallest eigenvalue is now zero for any T, $T'<T_c$. The overlap of the spin correlations at two different temperatures T, $T'<T_c$ is therefore infinitely long ranged, $\xi_{\Delta T}=\infty$, whatever the difference $\Delta T=T'-T$! (Note, however, that if, say, T' goes above $T_c=1$ then $\chi(Q)=1/T'<1$, and the spectrum develops a positive gap; there is no long-range overlap between the correlation functions in the spin glass and the paramagnet.)

It is easy to show that for T, $T'<T_c$ the leading infrared singularity of $\hat{G}_{\Delta T}(p)$ is

$$\hat{G}_{\Delta T}(p)\simeq\frac{\pi}{2}\frac{[Tc(T)T'c(T')]^{1/2}}{p^4} \qquad p\ll q_1, Q_1 \tag{16}$$

where $c(T)$ is the slope of $q(x)$ at $x=0$. The product $Tc(T)$ varies very little with T (it would be a constant if the Parisi–Toulouse (1980) hypothesis were exact). In the light of these results the order in the high-dimensional spin glass appears, in a sense, to be more robust against temperature changes than might have been expected.

Equation (11) and its T, T' counterpart are closely related to a particular component of the Gaussian propagator in the $H=0$, $T=T'$ system. This component was first calculated by Sompolinsky and Zippelius (1983) by a decoupling procedure in their dynamic approach, and later reproduced by Goltsev (1984a) via replicas. In De Dominicis and Kondor (1985) where the complete set of these propagators has been calculated, it was called $G_{11}^{00}(p)$. Recently a consistent static interpretation of these propagators has been given (Temesvàri *et al* 1988) and the propagator in question has been identified as the Fourier transform of the overlap $\overline{\langle S_iS_j\rangle^l\langle S_iS_j\rangle^{l'}}$ of correlations between pure states l, l' having zero overlap, $q_{ll'}=0$, between their local magnetisations.

With this we can then establish the following relationships:

$$\lim_{H=0}\overline{\langle S_iS_j\rangle_{H,T}\langle S_iS_j\rangle_{0,T}}=\lim_{T=T'}\overline{\langle S_iS_j\rangle_T\langle S_iS_j\rangle_{T'}}$$
$$=\overline{\langle S_iS_j\rangle_{0,T}^{l}\langle S_iS_j\rangle_{0,T}^{l'}}\qquad q_{ll'}=0. \tag{17}$$

These together with

$$\lim_{H=0}\overline{\langle S_i\rangle_{H,T}\langle S_i\rangle_{0,T}}=\lim_{T=T'}\overline{\langle S_i\rangle_{0,T}\langle S_i\rangle_{0,T'}}$$
$$=\overline{\langle S_i\rangle_{0,T}^{l}\langle S_i\rangle_{0,T}^{l'}}\qquad q_{l,l'}=0 \tag{18}$$

known from before, demonstrate the sensitivity of the spin glass to infinitesimal changes in the external parameters: the slightest change in H or T is enough to reshuffle the weights of the pure states, so that the magnetisation overlaps vanish, and of the various contributions to the correlation overlaps only those coming from the farthest states are left to survive. Given the current picture we have about spin glasses, equation (17) might well have been anticipated. What is surprising is the asymmetry we found between the finite H, respectively finite ΔT, cases. A finite H not only reshuffles the weights, but also reorganises the correlations $\langle S_iS_j\rangle_H$ sufficiently strongly to make their projection onto their old self $\langle S_iS_j\rangle_0$ vanish beyond a finite characteristic distance. In contrast to this, the correlations $\langle S_iS_j\rangle_{T'}$ keep a high degree of coherence with $\langle S_iS_j\rangle_T$ on any scale. (We note that in a field H, common to both systems, the overlap $\overline{\langle S_iS_j\rangle_{H,T}\langle S_iS_j\rangle_{H,T'}}$ becomes short ranged again.)

The persistence of the correlation overlaps under temperature changes brings back, admittedly in a strongly modified form, an old idea of Binder's (1977) who proposed the projection of the magnetisation pattern $\langle S_i\rangle_T$ onto that in the ground state as an order parameter. Although by (18) this projection vanishes identically, the long-range overlap between $\langle S_iS_j\rangle_T$ and $\langle S_iS_j\rangle_{T'}$ suggests that there may exist some underlying patterns after all. The projection of their weighted sum onto each other is washed away by the rapid change of the weights, as are most of the contributions to the projection of their correlations, but the approximate temperature independence of the remaining contributions shows that the structure of these patterns, as measured by their correlations, changes very little with temperature.

For T fixed and T' approaching T_c from below the long-range overlap between the correlations must eventually disappear. To see how this comes about we have to remember that the infrared singularity in $\hat{G}_{T,T'}(p)$ comes from the region where $p\ll Q_1$, and with $T'\to T_c-0$, $Q_1\sim(T_c-T')$ this region shrinks to zero. Put differently, for a fixed distance r and $T' \to T_c - 0$ we enter the region where $r \ll \xi \sim (T_c - T')^{-1}$ and $\hat{G}_{T,T'}(r)$ starts to drop quickly. The above results have been obtained in the Gaussian approximation around mean-field theory. As such, they have a direct relevance for the long-range model which, through its relation to optimisation and to the problem of associative memories (see Mézard *et al* 1987), has acquired a significance of its own. In view of the accumulating evidence for the inconsistency of the many-valley picture below $d = 6$ (Temesvàri *et al* 1988 and references therein), the relevance of our results for the $d = 3$ short-range spin glass is far less obvious. Nevertheless, by looking into the problem of chaos, we found interesting similarities to, and no less interesting differences from, the predictions of the phenomenological theory meant for the description of the $d = 3$ short-range system. I feel the exploration of the overlaps and discrepancies between the high-dimensional or long-range and the ($d =$ 3)-dimensional, short-range theories is well worth the effort and may help to better understand them both.

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